Math 142 Lecture 19 Notes

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1 Simplification of Cellular Decompositions

1.1 Removing S^2, $\mathbb{R}P^2$, or T^2

So far, we have shown that every connected, closed surface S is described by a cellular decomposition with a single polygon. We also said that a pair (P, ϕ) of a polygon and a gluing map corresponds to a word, where we read the labels of a word counterclockwise.

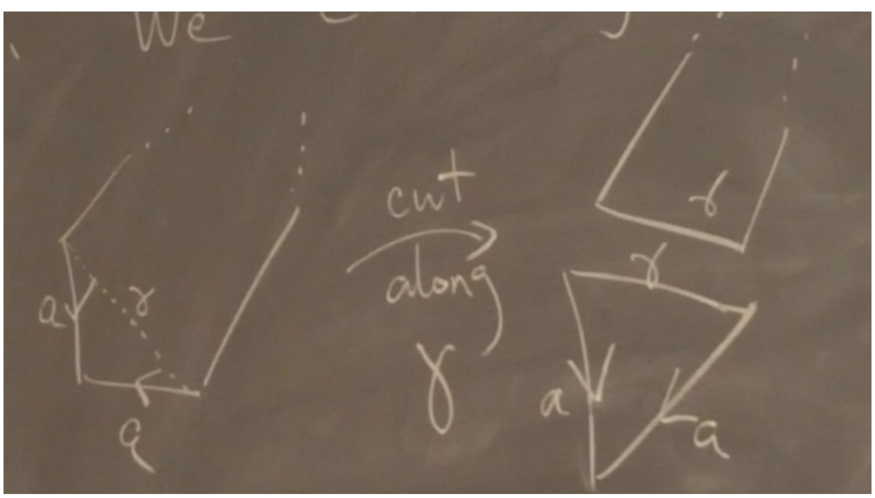
We also said that we can:

1. divide the polygon along a diagonal and reglue other edges,
2. flip a polygon over, and other homeomorphisms of $\mathbb{R}^2$.

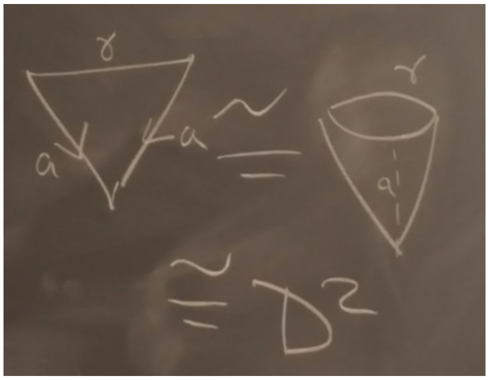
Let's continue the proof from last time.

Lemma 1.1. *If $S = Xaa^{-1}$, where $X \neq \varnothing$, then $S = X$.*

Proof. We cut along a loop γ in S.



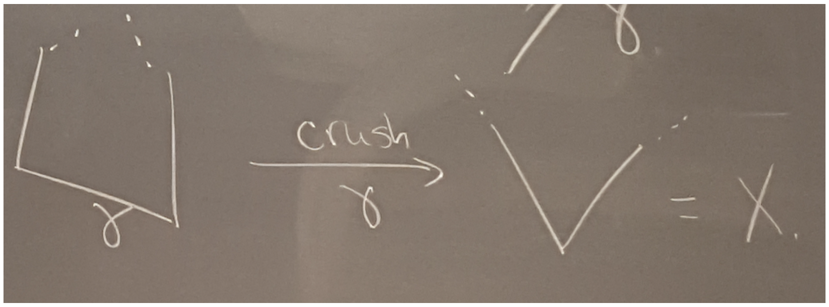
The triangle piece (call it S') is homeomorphic to a cone with circular boundary γ. This is homeomorphic to D^2.



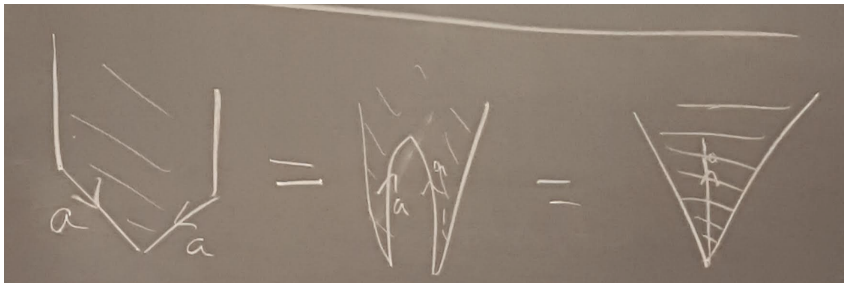
Notice that if we glue a D^2 to $S \setminus \mathrm{int}(S')$ and another D^2 to S', then we get:

1. from $S \setminus \mathrm{int}(S')$: S''
2. from S': S^2,

and $S''\#S^2 \cong S$. So by HW7 problem 3, $S'' \cong S$. Glueing a disc S^2 glued along γ is homeomorphic to $(S \setminus \mathrm{int}(S'))/\gamma$. This is



Alternatively, we can also think about it like moving in the vertex and folding the polygon in on itself.



$\square$

Lemma 1.2. *If $S = Xaa$, then $S \cong \mathbb{R}P^2$, where $S_1 = X$.*

Proof. This has the same proof as the previous lemma. $S \cong S_1 \# S_2$, where $S_2 = aa$. □

Lemma 1.3. *If $S = Xaba^{-1}b^{-1}$, then $S \cong T^2 \# S_1$, where $S_1 = X$.*

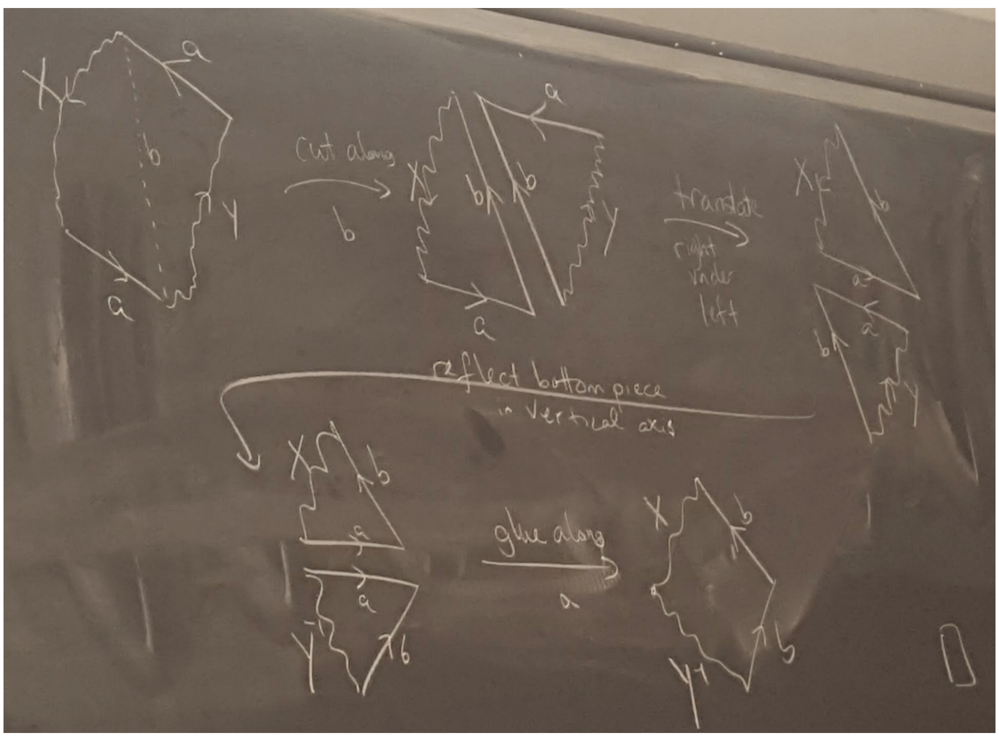
Proof. This has the same proof as, well. $S \cong S_1 \# S_2$, where $S_2 = aba^{-1}b^{-1}$. □

Remark 1.1. As you can see here, we have been omitting the actual homeomorphisms. It is expected that you should be able to come up with them yourself, given some time to think about it. In general, in mathematics, it is common to omit rigorous formalism when everyone involved is expected to be familiar enough with the concepts.[1]

1.2 Rearranging and decomposition of words

Lemma 1.4. *If $S = XaYa$, and $X, Y \neq \varnothing$, then $S = bbXY^{-1}$.*

Proof. Given our polygon, cut along b to get two pieces. Then translate the right hand piece under the left, and reflect it along a vertical axis to get the a edges pointing in the same direction. Then glue it back together.

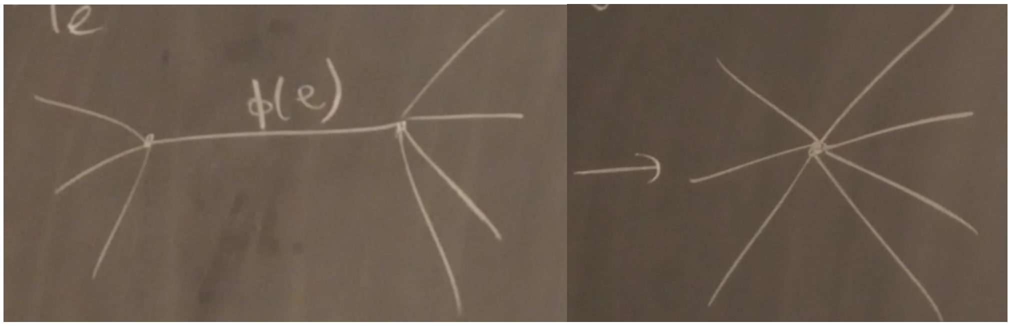


□

[1] This is Professor Conway's philosophy. Personally, I prefer to always prove statements in excruciating detail.

Lemma 1.5. *We can assume that $\phi(v) = \phi(v')$ for all vertices v, v' of P.*

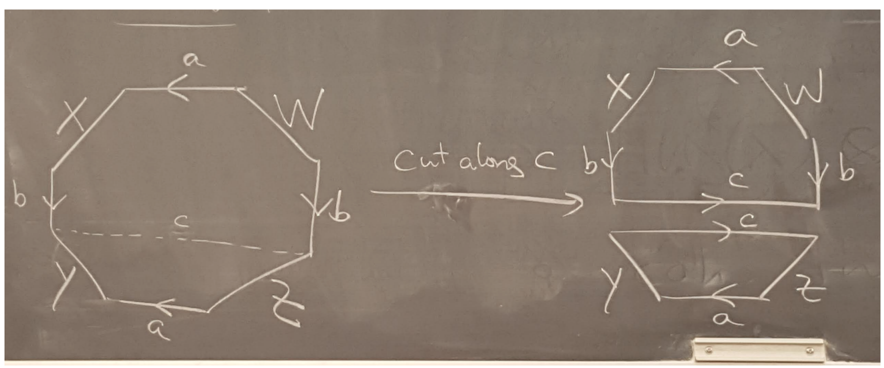
Proof. If v, v' are vertices connected by an edge e, and $\phi(v) \neq \phi(v')$, then $\phi|_e : e \to S$ is an embedding. Then we can contract $\phi(e)$ to a point.



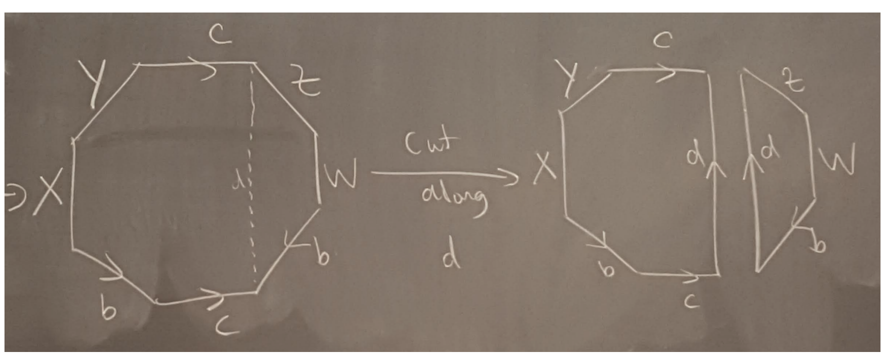
Check that we can construct (P', ϕ') from (P, ϕ) be contracting edge e and the corresponding edge e' with the same label and (P', ϕ') is a cellular decomposition of S. Then if all pairs of adjacent vertices in P satisfy $\phi(v) = \phi(v')$, then it is true for all vertices in P. $\square$

Lemma 1.6. *If $S = WaXbYa^{-1}Zb^{-1}$, then $S \cong T^2 \# S_1$, where $S_1 = ZYXW$.*

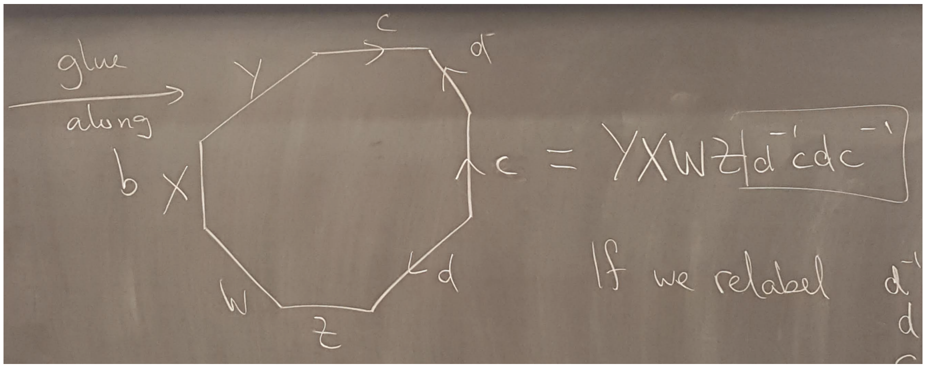
Proof. Given our polygon, cut along c to get two pieces.



Then glue along a, and then cut along d.



Then, if we glue along b, we get the word $YXWZd^{-1}cd^{-1}c$.



So our previous torus lemma gives us that $S \cong T^2 \# S_1$, where $S_1 = YXWZ = ZYXW$. $\square$

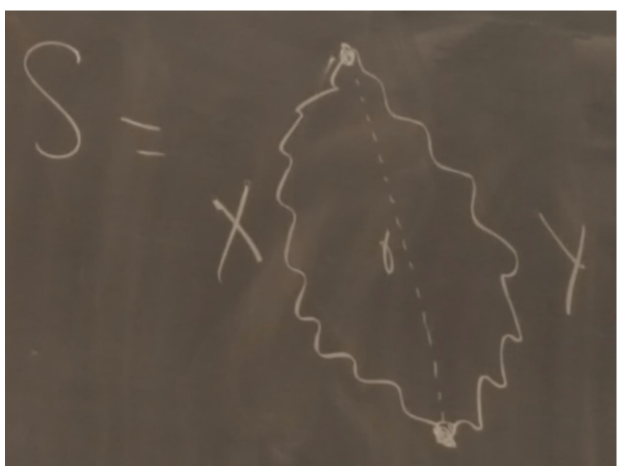
Theorem 1.1. *If S is a closed, connected surface, then*

$$S \cong S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{m} \# \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{n},$$

where m and n can equal 0.

Proof. Anytime we see $XaYa$, use two lemmas to write $S \cong S_1 \# \mathbb{R}P^2$. By repeating this, we write $S \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \# S_1$, where S_1 is described by a word X such that a and a^{-1} appear for every letter (and not adjacently); note that adjacent pairs aa^{-1} give us S^2, and connect summing with S^2 does nothing (unless S^2 is the only piece).

If there are no letters a and b such that $WaXbYa^{-1}Zb^{-1}$ appears, then we have $S = XY$, where the letters in X are disjoint from the letters in Y (check this yourself). So S looks like this:



By our vertex lemma, $\phi(\gamma)$ is a closed loop. If the letters of X and Y are disjoint, then γ separates S into 2 pieces. So similar to in our sphere lemma, $S \cong S_1 \# S_2$, where $S_1 = X$ and $S_2 = Y$. So either by this argument or by our previous lemmas, we write S as the connected sum of simpler pieces (T^2 or S_1, S_2). Eventually we will run out of letters and will have S^2 or T^2. $\square$